

Moderate deviations for recursive stochastic algorithms

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Abstract

We prove a moderate deviation principle for the continuous time interpolation of discrete time recursive stochastic processes. The methods of proof are somewhat different from the corresponding large deviation result, and in particular the proof of the upper bound is more complicated. The results can be applied to the design of accelerated Monte Carlo algorithms for certain problems, where schemes based on moderate deviations are easier to construct and in certain situations provide performance comparable to those based on large deviations.

1 Introduction

In this paper we consider \mathbb{R}^d -valued discrete time processes of the form

$$X_{i+1}^n = X_i^n + \frac{1}{n}b(X_i^n) + \frac{1}{n}v_i(X_i^n), \quad X_0^n = x_0,$$

where $\{v_i(\cdot)\}_{i \in \mathbb{N}_0}$ are zero mean random independent and identically distributed (iid) vector fields, and focus on their continuous time piecewise linear interpolations $\{X^n(t)\}_{0 \leq t \leq T}$ with $X^n(i/n) = X_i^n$ (see (2.5) for the precise definition). Under certain conditions there is a law of large number

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limit $X^0 \in C([0, T] : \mathbb{R}^d)$, and the large deviations of X^n from this limit have been studied extensively (see, e.g., [1, 8, 10, 13, 15]). Here we introduce a scaling $a(n)$ satisfying $a(n) \rightarrow 0$ and $a(n)\sqrt{n} \rightarrow \infty$, and study the amplified difference between X^n and its noiseless version $X^{n,0}$ (see Section 2 for the definition of $X^{n,0}$):

$$Y^n = a(n)\sqrt{n}(X^n - X^{n,0}).$$

Under Condition 2.1 introduced below $\sup_{t \in [0, T]} \|X^0(t) - X^{n,0}(t)\| \sim O(1/n)$, and hence this will behave the same asymptotically as $a(n)\sqrt{n}(X^n - X^0)$. We demonstrate, under weaker conditions on the noise $v_i(\cdot)$ than are necessary when considering X^n , that Y^n satisfies the large deviation principle on $C([0, T] : \mathbb{R}^d)$ with a ‘‘Gaussian’’ type rate function. As is customary for this type of scaling, we refer to this as moderate deviations.

To demonstrate this result we prove the equivalent Laplace principle, which involves evaluating limits of quantities of the form

$$a(n)^2 \log E \left[\exp \left\{ -\frac{1}{a(n)^2} F(Y^n) \right\} \right]$$

when F is bounded and continuous. This is done by representing each of these quantities in terms of a stochastic control problem, and then using weak convergence methods as in [10]. Key results needed in this approach are establishing tightness of controls and controlled processes, and identifying their limits.

While one might expect the proof of this moderate deviations result to be similar to the corresponding large deviations result, there are important differences. For example, the tightness proof is significantly more complicated in the case of moderate deviations than it is in the case of large deviations. For large deviations one is able to establish an a priori bound on certain relative entropy costs associated with any sequence of nearly minimizing controls, and under this boundedness of the relative entropy costs, the empirical measures of the controlled driving noises as well as the controlled processes are tight. However, owing to the scaling in moderate deviations, even with the information that the analogous relative entropy costs decay like $O(1/a(n)^2 n)$, tightness of the empirical measures of the noises does not hold. Instead, one must consider empirical measures of the conditional means of the noises, and additional effort is required for the law of large numbers type result that shows that the conditional means are adequate to determine the limit. This extra difficulty arises for moderate deviations (even with the vanishing relative entropy costs), because the noise itself is being amplified by $a(n)\sqrt{n}$.

A second way in which the proofs for large and moderate deviations differ is in their treatment of degenerate noise, i.e., problems where the support of $v_i(\cdot)$ is not all of \mathbb{R}^d . This leads to significant difficulties in the proof of the large deviation lower bound, and requires a delicate and involved mollification argument. In contrast, the proof in the setting of moderate deviations, though more involved than the nondegenerate case, is much more straightforward.

As a potential application of these results we mention their usefulness in the design and analysis of Monte Carlo schemes for events whose probability is small but not very small. For such problems the performance of standard Monte Carlo may not be adequate, especially if the quantity must be computed for many different parameter settings, as in say an optimization problem. Then accelerated Monte Carlo may be of interest, and as is well known such schemes (e.g., importance sampling and splitting) benefit by using information contained in the large deviation rate function as part of the algorithm design (e.g., [3, 6, 11, 12]). In a situation where one considers events of small but not too small probability one may find the moderate deviation approximation both adequate and relatively easy to apply, since moderate deviations lead to situations where the objects needed to design an efficient scheme can be explicitly constructed in terms of solutions to the linear-quadratic regulator. These issues will be explored elsewhere.

The existing literature on moderate deviations considers various settings. Baldi [2] considers the same scaling used here but with no state dependence. For the empirical measure of a Markov chain, de Acosta [5] and de Acosta and Chen [4] prove lower and upper bounds, respectively. Guillin [16] considers inhomogeneous functionals of a “fast” continuous time ergodic Markov chain, and in [17] this is extended to a small noise diffusion whose coefficients depend on the “fast” Markov chain. There are also results for martingale differences such as Dembo [7], Gao [14], and Djellout [9]. For various reasons, the issues previously mentioned regarding the difficulties in the proof of the upper bound and the simplification in the lower bound for degenerate noise do not play a role in these papers.

The paper is organized as follows. Section 2 gives the statement of the problem and notation. Section 3 contains the proof of tightness and the characterization of limits, which account for most of the mathematical difficulties, and are also the main results needed to prove the Laplace principle. Sections 4 and 5 give the proofs of the upper and lower Laplace bounds. Although all proofs are given for the time interval $[0, 1]$, they extend with only notational differences to $[0, T]$ for any $T \in (0, \infty)$.

2 Background and Notation

Let

$$X_{i+1}^n = X_i^n + \frac{1}{n}b(X_i^n) + \frac{1}{n}v_i(X_i^n), \quad X_0^n = x_0$$

where the $\{v_i(\cdot)\}_{i \in \mathbb{N}_0}$ are zero mean iid vector fields with distribution given by the stochastic kernel μ_x . Thus if $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ -algebra on \mathbb{R}^d , then $x \rightarrow \mu_x(B)$ is measurable for all $B \in \mathcal{B}(\mathbb{R}^d)$, $\mu_x(\cdot)$ is a probability measure on $\mathcal{B}(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$, and $P(v_i(x) \in B) = \mu_x(B)$ for all $x \in \mathbb{R}^d$, $B \in \mathcal{B}(\mathbb{R}^d)$ and $i \in \mathbb{N}_0$. Define

$$H_c(x, \alpha) \doteq \log \left(\int_{\mathbb{R}^d} e^{\langle y, \alpha \rangle} \mu_x(dy) \right)$$

for $\alpha \in \mathbb{R}^d$. The subscript c reflects the fact that this log moment generating function uses the centered distribution μ_x , rather than the usual $H(x, \alpha) = H_c(x, \alpha) + \langle \alpha, b(x) \rangle$. We will use the following.

Condition 2.1 • *There exists $\lambda > 0$ and $K_{mgf} < \infty$ such that*

$$\sup_{x \in \mathbb{R}^d} \sup_{\|\alpha\| \leq \lambda} H_c(x, \alpha) \leq K_{mgf}. \quad (2.1)$$

- $x \rightarrow \mu_x(dy)$ is continuous with respect to the topology of weak convergence.
- $b(x)$ is continuously differentiable, and the norm of both $b(x)$ and its derivative are uniformly bounded by some constant $K_b < \infty$.

Throughout this paper we let $\|\alpha\|_A^2 = \langle \alpha, A\alpha \rangle$ for any $\alpha \in \mathbb{R}^d$ and symmetric, nonnegative definite matrix A . Define

$$A_{ij}(x) \doteq \int_{\mathbb{R}^d} y_i y_j \mu_x(dy),$$

and note that the weak continuity of μ_x with respect to x and (2.1) ensure that $A(x)$ is continuous in x and its norm is uniformly bounded by some constant K_A . Note that

$$\frac{\partial H_c(x, 0)}{\partial \alpha_i} = \int_{\mathbb{R}^d} y_i \mu_x(dy) = 0$$

and

$$\frac{\partial^2 H_c(x, 0)}{\partial \alpha_i \partial \alpha_j} = \int_{\mathbb{R}^d} y_i y_j \mu_x(dy) = A_{ij}(x)$$

for all $i, j \in \{1, \dots, d\}$ and $x \in \mathbb{R}^d$, and that $A(x)$ is nonnegative-definite and symmetric. For $x \in \mathbb{R}^d$ we can therefore write

$$A(x) = Q(x) \Lambda(x) Q^T(x),$$

where $Q(x)$ is an orthogonal matrix whose columns are the eigenvectors of $A(x)$ and $\Lambda(x)$ is the diagonal matrix consisting of the eigenvalues of $A(x)$ in descending order. In what follows we define $\Lambda^{-1}(x)$ to be the diagonal matrix with diagonal entries equal to the inverse of the corresponding eigenvalue for the positive eigenvalues, and equal to ∞ for the zero eigenvalues. Then when we write

$$\|\alpha\|_{A^{-1}(x)}^2 = \|\alpha\|_{Q(x) \Lambda^{-1}(x) Q^T(x)}^2, \quad (2.2)$$

we mean a value of ∞ for $\alpha \in \mathbb{R}^d$ not in the linear span of the eigenvectors corresponding to the positive eigenvalues, and the standard value for vectors $\alpha \in \mathbb{R}^d$ in that linear span. Assumption (2.1) implies there exists some $K_{DA} < \infty$ and $\lambda_{DA} \in (0, \lambda]$ (independent of x) such that

$$\sup_{x \in \mathbb{R}^d} \sup_{\|\alpha\| \leq \lambda_{DA}} \max_{i,j,k} \left| \frac{\partial^3 H_c(x, \alpha)}{\partial \alpha_i \partial \alpha_j \partial \alpha_k} \right| \leq \frac{K_{DA}}{d^3}, \quad (2.3)$$

and consequently for all $\|\alpha\| \leq \lambda_{DA}$ and all $x \in \mathbb{R}^d$

$$\frac{1}{2} \|\alpha\|_{A(x)}^2 - \|\alpha\|^3 K_{DA} \leq H_c(x, \alpha) \leq \frac{1}{2} \|\alpha\|_{A(x)}^2 + \|\alpha\|^3 K_{DA}. \quad (2.4)$$

Define the continuous time linear interpolation of X_i^n by $X^n(i/n) = X_i^n$ for $i = 0, \dots, n$ and

$$X^n(t) = (i + 1 - nt)X_i^n + (nt - i)X_{i+1}^n \quad (2.5)$$

for $t \in (i/n, i/n + 1/n)$. In addition, define

$$X_{i+1}^{n,0} = X_i^{n,0} + \frac{1}{n} b(X_i^{n,0}), \quad X_0^{n,0} = x_0$$

and let $X^{n,0}(t)$ be the analogously defined continuous time linear interpolation. Clearly $X^{n,0}(t) \rightarrow X^0(t)$ in $C([0, 1] : \mathbb{R}^d)$, where

$$X^0(t) = \int_0^t b(X^0(s)) ds + x_0.$$

Since $Ev_i(x) = 0$ for all $x \in \mathbb{R}^d$, we know that $X^n(t) \rightarrow X^0(t)$ in $C([0, 1] : \mathbb{R}^d)$ in probability. One can estimate probabilities for events involving paths outside the law of large numbers limit X^0 by proving a large deviation principle and finding the corresponding rate function.

Definition 2.2 Let $\{Z^n, n \in \mathbb{N}\}$ be a sequence of random variables defined on a probability space (Ω, F, P) and taking values in a Polish space \mathcal{Z} . A function $I : \mathcal{Z} \rightarrow [0, \infty]$ is called a rate function if for any $M < \infty$ the set $\{x : I(x) \leq M\}$ is compact in \mathcal{Z} . The sequence $\{Z^n\}$ satisfies the large deviation principle on \mathcal{Z} with rate function I and sequence $r(n)$ if the following two conditions hold.

- *Large Deviation Upper Bound:* for each closed subset F of \mathcal{Z}

$$\limsup_{n \rightarrow \infty} r(n) \log P(Z^n \in F) \leq - \inf_{z \in F} I(z).$$

- *Large Deviation Lower Bound:* for each open subset G of \mathcal{Z}

$$\liminf_{n \rightarrow \infty} r(n) \log P(Z^n \in G) \geq - \inf_{z \in G} I(z).$$

Under significantly stronger assumptions, including the assumption

$$\sup_{x \in \mathbb{R}^d} \sup_{\alpha \in \mathbb{R}^d} H_c(x, \alpha) < \infty,$$

it has been shown that $X^n(t)$ satisfies the large deviation principle on $C([0, 1] : \mathbb{R}^d)$ with sequence $r(n) = 1/n$ and rate function

$$I_L(\phi) = \inf \left\{ \int_0^1 L_c(\phi(s), u(s)) ds : \phi(t) = x_0 + \int_0^t b(\phi(s)) ds + \int_0^t u(s) ds, t \in [0, 1] \right\}.$$

where

$$L_c(x, \beta) = \sup_{\alpha \in \mathbb{R}^d} \{\langle \alpha, \beta \rangle - H_c(x, \alpha)\}$$

is the Legendre transform of $H_c(x, \alpha)$ [10, 19, 20, 21, 22].

Assume $a(n)$ satisfies

$$a(n) \rightarrow 0 \text{ and } a(n)\sqrt{n} \rightarrow \infty. \quad (2.6)$$

We define the rescaled difference

$$Y^n(t) = a(n)\sqrt{n}(X^n(t) - X^{n,0}(t)).$$

As noted in the introduction, the result stated below also holds with the interval $[0, 1]$ replaced by $[0, T]$, $T \in (0, \infty)$. Let D denote the gradient operator.

Theorem 2.3 *Assume Condition 2.1. Then $\{Y^n\}_{n \in \mathbb{N}}$ satisfies the large deviation principle on $C([0, 1] : \mathbb{R}^d)$ with sequence $a(n)^2$ and rate function*

$$I_M(\phi) = \inf \left\{ \frac{1}{2} \int_0^1 \|u(t)\|^2 dt : \phi(t) = \int_0^t Db(X^0(s))\phi(s)ds + \int_0^t A^{1/2}(X^0(s))u(s)ds, t \in [0, 1] \right\}.$$

I_M is essentially the same as what one would obtain by using a linear approximation around the law of large numbers limit X^0 of the dynamics and a quadratic approximation of the costs in I_L . To prove the LDP, it suffices to show the Laplace principle [10, Theorem 1.2.3]

$$\begin{aligned} & \lim_{n \rightarrow \infty} -a(n)^2 \log E \left[e^{-\frac{1}{a(n)^2} F(Y^n)} \right] \\ &= \inf_{u \in L^2([0, 1]; \mathbb{R}^d)} \left\{ \frac{1}{2} \int_0^1 \|u(s)\|^2 ds + F\left(\phi^{A^{1/2}(X^0)u}\right) \right\} \end{aligned} \quad (2.7)$$

where

$$\phi^u(t) = \int_0^t Db(X^0(s))\phi^u(s)ds + \int_0^t u(s)ds. \quad (2.8)$$

Note that

$$Y_{i+1}^n = Y_i^n + \frac{a(n)}{\sqrt{n}} \left(b(X_i^n) - b(X_i^{n,0}) \right) + \frac{a(n)}{\sqrt{n}} v_i(X_i^n), \quad Y_0^n = 0$$

For $\eta, \mu \in \mathcal{P}(\mathbb{R}^d)$ [the set of probability measures on $\mathcal{B}(\mathbb{R}^d)$], the relative entropy of η with respect to μ is defined by

$$R(\eta \| \mu) \doteq \int_{\mathbb{R}^d} \log \left(\frac{d\eta}{d\mu}(x) \right) \eta(dx) \in [0, \infty]$$

if η is absolutely continuous with respect to μ , and $R(\eta \| \mu) \doteq \infty$ otherwise. For general properties of relative entropy we refer to [10, Section 1.4]. The

variational formula [10, Proposition 1.4.2(a)] and chain rule [10, Theorem C.3.1] imply that

$$-a(n)^2 \log E \left[e^{-\frac{1}{a(n)^2} F(Y^n)} \right] = \inf_{\eta} E \left[\sum_{i=0}^{n-1} a(n)^2 R(\eta_i \| \mu_{\bar{X}_i^n}) + F(\bar{Y}^n) \right] \quad (2.9)$$

for any bounded, continuous $F : C([0, 1] : \mathbb{R}^d) \rightarrow \mathbb{R}$. Here $\eta \in \mathcal{P}((\mathbb{R}^d)^n)$ is the joint distribution of $(\bar{v}_0, \dots, \bar{v}_{n-1})$, $\eta_i(\cdot)$ is the conditional distribution on \bar{v}_i given $(\bar{v}_0, \dots, \bar{v}_{i-1})$,

$$\bar{X}_{i+1}^n = \bar{X}_i^n + \frac{1}{n} b(\bar{X}_i^n) + \frac{1}{n} \bar{v}_i, \quad \bar{X}_0^n = x_0, \quad (2.10)$$

$$\bar{Y}_{i+1}^n = \bar{Y}_i^n + \frac{a(n)}{\sqrt{n}} \left(b(\bar{X}_i^n) - b(X_i^{n,0}) \right) + \frac{a(n)}{\sqrt{n}} \bar{v}_i, \quad \bar{Y}_0^n = 0 \quad (2.11)$$

and, similar to (2.5), $\bar{X}^n(t)$ and $\bar{Y}^n(t)$ are the continuous time linear interpolations of $\{\bar{X}_i^n\}_{i=0, \dots, n}$ and $\{\bar{Y}_i^n\}_{i=0, \dots, n}$. Note that η_i depends on past values of the noise, but we suppress this dependence in the notation. We will prove (2.7) by proving the lower bound

$$\begin{aligned} & \liminf_{n \rightarrow \infty} -a(n)^2 \log E \left[e^{-\frac{1}{a(n)^2} F(Y^n)} \right] \\ & \geq \inf_{u \in L^2([0,1]: \mathbb{R}^d)} \left\{ \frac{1}{2} \int_0^1 \|u(s)\|^2 ds + F\left(\phi^{A^{1/2}(X^0)} u\right) \right\} \end{aligned} \quad (2.12)$$

and the upper bound

$$\begin{aligned} & \limsup_{n \rightarrow \infty} -a(n)^2 \log E \left[e^{-\frac{1}{a(n)^2} F(Y^n)} \right] \\ & \leq \inf_{u \in L^2([0,1]: \mathbb{R}^d)} \left\{ \frac{1}{2} \int_0^1 \|u(s)\|^2 ds + F\left(\phi^{A^{1/2}(X^0)} u\right) \right\}. \end{aligned} \quad (2.13)$$

We will use a tightness and weak convergence result in the proofs of both of these bounds, but first establish notation used in the rest of the paper.

Construction 2.4 Given a sequence of measures $\{\eta^n\}_{n \in \mathbb{N}}$ with each $\eta^n \in \mathcal{P}((\mathbb{R}^d)^n)$, define the following. Let $(\bar{v}_0^n, \dots, \bar{v}_{n-1}^n)$ be random variables with distribution η^n , and define $\{\bar{X}_i^n\}_{i=0, \dots, n}$ and $\{\bar{Y}_i^n\}_{i=0, \dots, n}$ by (2.10) and (2.11). Let

$$\bar{X}^n(t) \doteq (i+1-nt)\bar{X}_i^n + (nt-i)\bar{X}_{i+1}^n$$

and

$$\bar{Y}^n(t) \doteq (i+1-nt)\bar{Y}_i^n + (nt-i)\bar{Y}_{i+1}^n$$

for $t \in [i/n, (i+1)/n]$, $i = 0, \dots, n-1$ be their continuous time linear interpolations. Define the conditional means of the noises

$$w^n(t) \doteq \int_{\mathbb{R}^d} y \eta_i^n(dy) \text{ for } t \in \left[\frac{i}{n}, \frac{i+1}{n} \right),$$

the amplified conditional means

$$\hat{w}^n(t) \doteq a(n)\sqrt{n}w^n(t),$$

and random measures on $\mathbb{R}^d \otimes [0, 1]$ by

$$\hat{\eta}^n(dy \otimes dt) \doteq \delta_{\hat{w}^n(t)}(dy)dt = \delta_{a(n)\sqrt{n}w^n(t)}(dy)dt.$$

We will refer to this construction when given η^n to identify associated $\bar{X}^n, \bar{Y}^n, \hat{w}^n$ and $\hat{\eta}^n$. Given $\nu \in \mathcal{P}(E_1 \times E_2)$, with each $E_i, i = 1, 2$ a Polish space, let ν_2 denote the second marginal of ν , and let $\nu_{1|2}$ denote the conditional distribution on E_1 given a point in E_2 .

Theorem 2.5 *Let $\{\eta^n\}$ be a sequence of measures, each $\eta^n \in \mathcal{P}((\mathbb{R}^d)^n)$, and define the corresponding random variables as in Construction 2.4. Assume that for some $K_E < \infty$*

$$\sup_{n \in \mathbb{N}} \left\{ a(n)^2 n E \left[\frac{1}{n} \sum_{i=0}^{n-1} R(\eta_i^n \| \mu_{\bar{X}_i^n}) \right] \right\} \leq K_E. \quad (2.14)$$

Then $\{(\hat{\eta}^n, \bar{Y}^n)\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(\mathbb{R}^d \otimes [0, 1]) \otimes C([0, 1] : \mathbb{R}^d)$. Consider a subsequence (keeping the index n for convenience) such that $\{(\hat{\eta}^n, \bar{Y}^n)\}$ converges weakly to $(\hat{\eta}, \hat{Y})$. Then with probability 1 $\hat{\eta}_2(dt)$ is Lebesgue measure and

$$\hat{Y}(t) = \int_0^t Db(X^0(s))\hat{Y}(s)ds + \int_0^t \hat{w}(s)ds, \quad (2.15)$$

where

$$\hat{w}(t) = \int_{\mathbb{R}^d} y \hat{\eta}_{1|2}(dy | t).$$

In addition,

$$\liminf_{n \rightarrow \infty} a(n)^2 n E \left[\frac{1}{n} \sum_{i=0}^{n-1} R(\eta_i^n \| \mu_{\bar{X}_i^n}) \right] \geq E \left[\int_0^1 \frac{1}{2} \|\hat{w}(s)\|_{A^{-1}(X^0(s))}^2 ds \right]. \quad (2.16)$$

3 Proof of Theorem 2.5

Assume that the bound (2.14) holds. We will show tightness of the $\{\hat{\eta}^n\}$ measures using the following lemma.

Lemma 3.1 *Assume Condition 2.1 and let*

$$L_c(x, \beta) = \sup_{\alpha \in \mathbb{R}^d} \{\langle \alpha, \beta \rangle - H_c(x, \alpha)\} \quad (3.1)$$

be the Legendre transform of $H_c(x, \cdot)$. Then for any $x \in \mathbb{R}^d$ and $\eta \in \mathcal{P}(\mathbb{R}^d)$

$$R(\eta \| \mu_x) \geq L_c \left(x, \int_{\mathbb{R}^d} y \eta(dy) \right).$$

Proof. While the result is likely known we could not locate a proof (see [10, Lemma 6.2.3(f)] for a proof when $H_c(x, \alpha)$ is finite for all $\alpha \in \mathbb{R}^d$), and so for completeness provide the details. If $R(\eta \| \mu_x) = \infty$ the lemma is automatically true, so we assume $R(\eta \| \mu_x) < \infty$. Define $\ell(b) \doteq b \log b - b + 1$ and note that for $a, b \geq 0$

$$ab \leq e^a + \ell(b). \quad (3.2)$$

From (2.1) we have

$$\int_{\mathbb{R}^d} e^{\frac{\lambda}{2d} \|y\|} \mu_x(dy) \leq 2^d e^{dK_{\text{mgf}}} < \infty.$$

Therefore

$$\begin{aligned} & \int_{\mathbb{R}^d} \frac{\lambda}{2d} \|y\| \frac{d\eta}{d\mu_x}(y) \mu_x(dy) \\ & \leq \int_{\mathbb{R}^d} e^{\frac{\lambda}{2d} \|y\|} \mu_x(dy) + \int_{\mathbb{R}^d} \ell \left(\frac{d\eta}{d\mu}(y) \right) \mu_x(dy) \\ & \leq 2^d e^{dK_{\text{mgf}}} + R(\eta \| \mu_x), \end{aligned}$$

and consequently for any $\alpha \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \|\alpha\| \|y\| \frac{d\eta}{d\mu_x}(y) \mu_x(dy) \leq \frac{2^d \|\alpha\|}{\lambda} \left(2^d e^{dK_{\text{mgf}}} + R(\eta \| \mu_x) \right) < \infty. \quad (3.3)$$

Define the bounded, continuous function

$$F_K(y, \alpha) = \begin{cases} \langle \alpha, y \rangle & \text{if } |\langle \alpha, y \rangle| \leq K \\ \frac{K \langle \alpha, y \rangle}{|\langle \alpha, y \rangle|} & \text{otherwise,} \end{cases}$$

and note that (3.3) and dominated convergence give

$$\lim_{K \rightarrow \infty} \int_{\mathbb{R}^d} F_K(y, \alpha) \eta(dy) = \left\langle \alpha, \int_{\mathbb{R}^d} y \eta(dy) \right\rangle.$$

In addition, dominated convergence gives

$$\lim_{K \rightarrow \infty} \int_{\{y: \langle \alpha, y \rangle < 0\}} e^{F_K(y, \alpha)} \mu_x(dy) = \int_{\{y: \langle \alpha, y \rangle < 0\}} e^{\langle \alpha, y \rangle} \mu_x(dy)$$

and monotone convergence gives

$$\lim_{K \rightarrow \infty} \int_{\{y: \langle \alpha, y \rangle \geq 0\}} e^{F_K(y, \alpha)} \mu_x(dy) = \int_{\{y: \langle \alpha, y \rangle \geq 0\}} e^{\langle \alpha, y \rangle} \mu_x(dy),$$

so

$$\lim_{K \rightarrow \infty} \log \left(\int_{\mathbb{R}^d} e^{F_K(y, \alpha)} \mu_x(dy) \right) = H_c(x, \alpha).$$

By the Donsker-Varadhan variational formula [10, Lemma 1.4.3(a)]

$$R(\eta \| \mu_x) \geq \int_{\mathbb{R}^d} F_K(y, \alpha) \eta(dy) - \log \left(\int_{\mathbb{R}^d} e^{F_K(y, \alpha)} \mu_x(dy) \right)$$

for all $K < \infty$ and $\alpha \in \mathbb{R}^d$, and so

$$R(\eta \| \mu_x) \geq \sup_{\alpha \in \mathbb{R}^d} \left\{ \left\langle \alpha, \int_{\mathbb{R}^d} y \eta(dy) \right\rangle - H_c(x, \alpha) \right\} = L_c \left(x, \int_{\mathbb{R}^d} y \eta(dy) \right),$$

which completes the proof of the lemma. ■

The lemma implies the following theorem, which in turn will give tightness of $\{\hat{\eta}^n\}$.

Theorem 3.2 *Assume Condition 2.1 and (2.14). For the processes $\{w^n\}$ obtained in Construction 2.4*

$$\sup_{n \in \mathbb{N}} E \left[\int_0^1 a(n) \sqrt{n} \|w^n(s)\| ds \right] < \infty.$$

In addition, $\{a(n) \sqrt{n} w^n(\cdot)\}_{n \in \mathbb{N}}$ is uniformly integrable in the sense that

$$\lim_{C \rightarrow \infty} \limsup_{n \rightarrow \infty} E \left[\int_0^1 1_{\{a(n) \sqrt{n} \|w^n(s)\| > C\}} a(n) \sqrt{n} \|w^n(s)\| ds \right] = 0.$$

Proof. We use the following inequality. Let $G > 0$ satisfy $\lambda_{DA} \min_{n \in \mathbb{N}} \{a(n) \sqrt{n}\} = \sqrt{G}$ [recall (2.6)] so that $\lambda_{DA} \geq \sqrt{G}/a(n) \sqrt{n}$ for all n . Define L_c by (3.1). Let $\bar{K} \doteq \lambda_{DA} K_{DA} + K_A/2$. Then with e_i denoting the standard unit vectors

$$\begin{aligned} & a(n)^2 n L_c(x, \beta) \\ &= \sup_{\alpha \in \mathbb{R}^d} [a(n) \sqrt{n} \langle \alpha, a(n) \sqrt{n} \beta \rangle - a(n)^2 n H_c(x, \alpha)] \\ &\geq \pm a(n) \sqrt{n} \left\langle \frac{\sqrt{G}}{a(n) \sqrt{n}} e_i, a(n) \sqrt{n} \beta \right\rangle - a(n)^2 n H_c \left(x, \pm \frac{\sqrt{G}}{a(n) \sqrt{n}} e_i \right) \\ &\geq \pm \sqrt{G} a(n) \sqrt{n} \beta_i - \frac{1}{2} G \|A(x)\| - G \lambda_{DA} K_{DA} \\ &\geq \pm \sqrt{G} a(n) \sqrt{n} \beta_i - G \bar{K}, \end{aligned}$$

where the first inequality follows from making a specific choice of α and the second uses (2.4). Therefore

$$da(n)^2 n L_c(x, \beta) + dG \bar{K} \geq \sqrt{G} a(n) \sqrt{n} \|\beta\|. \quad (3.4)$$

Using the bound on L_c from Lemma 3.1 together with (2.14),

$$\begin{aligned} & d \left(\frac{K_E}{\sqrt{G}} + \sqrt{G} \bar{K} \right) \\ &\geq \frac{da(n)^2 n}{\sqrt{G}} E \left[\int_0^1 L_c \left(\bar{X}^n \left(\frac{\lfloor ns \rfloor}{n} \right), w^n(s) \right) ds \right] + d\sqrt{G} \bar{K} \\ &\geq E \left[\int_0^1 a(n) \sqrt{n} \|w^n(s)\| ds \right]. \end{aligned} \quad (3.5)$$

For the uniform integrability, let $C \in (1, \infty)$ be arbitrary and consider n large enough that

$$\min\{\lambda_{DA}, 1\} \geq \frac{\sqrt{C}}{a(n)\sqrt{n}}.$$

Since $\lambda_{DA} \geq 1/a(n)\sqrt{n}$ (which corresponds to using the estimate above with $G = 1$) we have

$$E \left[\int_0^1 a(n)\sqrt{n} \|w^n(s)\| ds \right] \leq K^* \doteq d \left(K_E + \frac{1}{2}K_A + \lambda_{DA}K_{DA} \right).$$

Therefore

$$E \left[\int_0^1 1_{\{a(n)\sqrt{n}\|w^n(s)\| > C\}} ds \right] \leq \frac{K^*}{C},$$

which combined with the estimate (3.4) with G replaced by C and (3.5) gives

$$\begin{aligned} & \sqrt{C} E \left[\int_0^1 1_{\{a(n)\sqrt{n}\|w^n(s)\| > C\}} a(n)\sqrt{n} \|w^n(s)\| ds \right] \\ & \leq E \left[d \int_0^1 1_{\{a(n)\sqrt{n}\|w^n(s)\| > C\}} \left(a(n)^2 n L_c \left(\bar{X}^n \left(\frac{\lfloor ns \rfloor}{n} \right), w^n(s) \right) + C \bar{K} \right) ds \right] \\ & \leq da(n)^2 n E \left[\int_0^1 L_c \left(\bar{X}^n \left(\frac{\lfloor ns \rfloor}{n} \right), w^n(s) \right) ds \right] \\ & \quad + Cd\bar{K} E \left[\int_0^1 1_{\{a(n)\sqrt{n}\|w^n(s)\| > C\}} ds \right] \\ & \leq K^* d (1 + \bar{K}). \end{aligned}$$

We conclude that

$$\lim_{C \rightarrow \infty} \limsup_{n \rightarrow \infty} E \left[\int_0^1 1_{\{a(n)\sqrt{n}\|w^n(s)\| > C\}} a(n)\sqrt{n} \|w^n(s)\| ds \right] = 0,$$

which is the claimed uniform integrability. ■

We continue with the proof of Theorem 2.5. Note that $g(y, t) = \|y\|$ is a tightness function on $\mathbb{R}^d \otimes [0, 1]$, so by [10, Theorem A.3.17]

$$G(\eta) = \int_{\mathbb{R}^d \otimes [0, 1]} \|y\| \eta(dy \otimes dt)$$

is a tightness function on $\mathcal{P}(\mathbb{R}^d \otimes [0, 1])$ and

$$\bar{G}(\gamma) = \int_{\mathcal{P}(\mathbb{R}^d \otimes [0, 1])} \int_{\mathbb{R}^d \otimes [0, 1]} \|y\| \eta(dy \otimes dt) \gamma(d\eta)$$

is a tightness function on $\mathcal{P}(\mathcal{P}(\mathbb{R}^d \otimes [0, 1]))$. Since

$$\sup_{n \in \mathbb{N}} EG(\hat{\eta}^n) = \sup_{n \in \mathbb{N}} E \left[\int \|y\| \hat{\eta}^n(dy \otimes dt) \right] = \sup_{n \in \mathbb{N}} E \left[\int_0^1 a(n) \sqrt{n} \|w^n(s)\| ds \right] < \infty,$$

$\{\hat{\eta}^n\}$ is tight and consequently there is a subsequence of $\{\hat{\eta}^n\}$ which converges weakly. To simplify notation we retain n as the index of this convergent subsequence, and denote the weak limit of $\{\hat{\eta}^n\}$ by $\hat{\eta}$. Note that for all n the second marginal of $\hat{\eta}^n(dy \otimes dt)$, which we denote by $\hat{\eta}_2^n(dt)$, is Lebesgue measure, and therefore $\hat{\eta}_2(dt)$ is Lebesgue measure with probability 1.

Our aim is to show that $\bar{Y}^n(t) \rightarrow \hat{Y}(t)$ weakly in $C([0, 1] : \mathbb{R}^d)$, where $\hat{Y}(t)$ is given by (2.15) in terms of the weak limit $\hat{\eta}$. To achieve this we introduce the following processes which serve as intermediate steps. Let $\check{Y}_0^n = 0$ and

$$\check{Y}_{i+1}^n = \check{Y}_i^n + \frac{a(n)}{\sqrt{n}} \left(b \left(X_i^{n,0} + \frac{1}{a(n)\sqrt{n}} \check{Y}_i^n \right) - b \left(X_i^{n,0} \right) \right) + \frac{a(n)}{\sqrt{n}} w^n \left(\frac{i}{n} \right),$$

together with its continuous time linear interpolation defined for $t \in [i/n, i/n + 1/n]$ by

$$\check{Y}^n(t) = (i + 1 - nt) \check{Y}_i^n + (nt - i) \check{Y}_{i+1}^n.$$

Also let

$$\hat{Y}^n(t) = \int_0^t Db(X^0(s)) \hat{Y}^n(s) ds + \int_0^t \hat{w}^n(s) ds \quad (3.6)$$

where

$$\hat{w}^n(t) = \int_{\mathbb{R}^d} y \hat{\eta}_{1|2}^n(dy | t)$$

as in Construction 2.4. These are both random variables taking values in $C([0, 1] : \mathbb{R}^d)$. Note that \bar{Y}^n differs from \check{Y}^n because \bar{Y}^n is driven by the actual noises and \check{Y}^n is driven by their conditional means. While the driving terms of \hat{Y}^n and \check{Y}^n are the same [recall that $a(n)\sqrt{n}w^n(t) = \hat{w}^n(t)$], they differ in that \check{Y}^n is still a linear interpolation of a discrete time process whereas \hat{Y}^n satisfies an ODE. The goal is to show that along the subsequence where $\hat{\eta}^n \rightarrow \hat{\eta}$ weakly

$$\bar{Y}^n - \check{Y}^n \rightarrow 0, \check{Y}^n - \hat{Y}^n \rightarrow 0, \text{ and } \hat{Y}^n \rightarrow \hat{Y}$$

in $C([0, 1] : \mathbb{R}^d)$, all in distribution. To show $\hat{Y}^n \rightarrow \hat{Y}$ we show that $\{\hat{Y}^n\}$ is tight in $C([0, 1] : \mathbb{R}^d)$ and use the mapping defined by (3.6) from $\int_0^\cdot \hat{w}^n$ to \hat{Y}^n . Recall that $\sup_{x \in \mathbb{R}^d} \|Db(x)\| \leq K_b$. The following lemma is an easy consequence of Gronwall's inequality.

Lemma 3.3 *Let $u \in L^1([0, 1] : \mathbb{R}^d)$ be arbitrary and ϕ^u be defined as in (2.8). Then for $0 \leq s \leq t \leq 1$*

$$\|\phi^u(t) - \phi^u(s)\| \leq (t - s)K_b e^{K_b} \int_0^1 \|u(r)\| dr + \int_s^t \|u(r)\| dr.$$

With this lemma and the uniform integrability of $\{\hat{\eta}^n\}$ given in Theorem 3.2, tightness follows.

Lemma 3.4 *Assume Condition 2.1 and (2.14). The sequence $\{\hat{Y}^n\}$ defined in (3.6) in terms of the measures $\{\eta^n\}$ via Construction 2.4 is tight in $C([0, 1] : \mathbb{R}^d)$, as is $\{\int_0^\cdot \hat{w}^n ds\}$.*

Proof. It suffices to show that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\limsup_{n \rightarrow \infty} P \left(\sup_{|s-t| \leq \delta} \|\hat{Y}^n(t) - \hat{Y}^n(s)\| > \varepsilon \right) < \varepsilon.$$

Define

$$\begin{aligned} T(C) &\doteq \limsup_{n \rightarrow \infty} E \left[\int_0^1 1_{\{\|\hat{w}^n(t)\| > C\}} \|\hat{w}^n(t)\| dt \right] \\ &= \limsup_{n \rightarrow \infty} E \left[\int_{\{\|y\| > C\}} \|y\| \hat{\eta}^n(dy \otimes dt) \right]. \end{aligned}$$

By Theorem 3.2 $T(C) \rightarrow 0$ as $C \rightarrow \infty$. Define also $K_\eta = \sup_{n \in \mathbb{N}} E \int_0^1 \|\hat{w}^n(t)\| dt$, which is finite by Theorem 3.2. Let $\varepsilon > 0$ be arbitrary. Then for any $s < t$ satisfying $t - s \leq \delta$ the previous lemma implies

$$\|\hat{Y}^n(t) - \hat{Y}^n(s)\| \leq \delta K_b e^{K_b} \int_0^1 \|\hat{w}^n(r)\| dr + \int_s^t \|\hat{w}^n(r)\| dr.$$

Since

$$\int_s^t \|\hat{w}^n(r)\| dr \leq C\delta + \int_0^1 1_{\{\|\hat{w}^n(r)\| > C\}} \|\hat{w}^n(r)\| dr,$$

it follows that

$$\begin{aligned} \left\| \hat{Y}^n(t) - \hat{Y}^n(s) \right\| &\leq \delta \left(C + K_b e^{K_b} \int_0^1 \|\hat{w}^n(r)\| dr \right) \\ &\quad + \int_0^1 1_{\{\|\hat{w}^n(r)\| > C\}} \|\hat{w}^n(r)\| dr. \end{aligned}$$

Hence by Markov's inequality

$$\begin{aligned} \limsup_{n \rightarrow \infty} P \left(\sup_{|s-t| \leq \delta} \left\| \hat{Y}^n(t) - \hat{Y}^n(s) \right\| > \varepsilon \right) \\ \leq \frac{\delta}{\varepsilon} \limsup_{n \rightarrow \infty} E \left[\left(C + K_b e^{K_b} \int_0^1 \|\hat{w}^n(r)\| dr \right) \right] \\ + \frac{1}{\varepsilon} \limsup_{n \rightarrow \infty} E \left[\int_0^1 1_{\{\|\hat{w}^n(r)\| > C\}} \|\hat{w}^n(r)\| dr \right] \\ \leq \frac{\delta}{\varepsilon} (C + K_b e^{K_b} K_\eta) + \frac{1}{\varepsilon} T(C). \end{aligned}$$

Choose $C < \infty$ such that $T(C) < \varepsilon^2/2$ and then choose $\delta > 0$ so that the $\delta(C + K_b e^{K_b} K_\eta) < \varepsilon^2/2$. This shows the tightness of $\{\hat{Y}^n\}$. The tightness of $\{\int_0^\cdot \hat{w}^n ds\}$ is simpler, and follows from the bound

$$\limsup_{n \rightarrow \infty} P \left(\sup_{|s-t| \leq \delta} \int_s^t \|\hat{w}^n(r)\| dr > \varepsilon \right) \leq \delta \frac{C}{\varepsilon} + \frac{1}{\varepsilon} T(C).$$

■

We still need to show that \hat{Y}^n converges to \hat{Y} . This also relies on the uniform integrability given by Theorem 3.2.

Lemma 3.5 *Assume Condition 2.1 and (2.14). Let the sequence $\{\hat{Y}^n(t)\}$ be defined by (3.6), let $\hat{Y}(t)$ be defined by (2.15), and consider a convergent subsequence $\{(\hat{Y}^n, \hat{\eta}^n)\}$ with limit $(\hat{Y}^*, \hat{\eta})$. Then w.p.1 $\hat{Y}^* = \hat{Y}$.*

Proof. We can write

$$\hat{Y}^n(t) = \int_0^t Db(X^0(s)) \hat{Y}^n(s) ds + \int_0^t \int_{\mathbb{R}^d} y \hat{\eta}^n(dy \otimes ds).$$

Using the uniform integrability proved in Theorem 3.2 and that $\hat{\eta}_2$ is Lebesgue measure w.p.1, sending $n \rightarrow \infty$ and using the definition of \hat{w} gives

$$\begin{aligned}\hat{Y}^*(t) &= \int_0^t Db(X^0(s))\hat{Y}^*(s)ds + \int_0^t \int_{\mathbb{R}^d} y\hat{\eta}(dy \otimes ds) \\ &= \int_0^t Db(X^0(s))\hat{Y}^*(s)ds + \int_0^t \hat{w}(s)ds.\end{aligned}$$

By uniqueness of the solution, $\hat{Y}^* = \hat{Y}$ follows. ■

It remains to show $\bar{Y}^n - \check{Y}^n \rightarrow 0$ and $\check{Y}^n - \hat{Y}^n \rightarrow 0$. We begin with $\bar{Y}^n - \check{Y}^n \rightarrow 0$. Recall that the difference between \bar{Y}^n and \check{Y}^n is that the first is driven by the actual noises and the second is driven by their conditional means. The following theorem is a law of large numbers type result for the difference between the noises and their conditional means, and is the most complicated part of the analysis.

Theorem 3.6 *Assume Condition 2.1 and (2.14). Consider the sequence $\{\bar{v}_i^n\}_{i=0,\dots,n-1}$ of controlled noises and $\{w^n(i/n)\}_{i=0,\dots,n-1}$ of means of the controlled noises as in Construction 2.4. For $i \in \{1, \dots, n\}$ let*

$$W_i^n \doteq \frac{1}{n} \sum_{j=0}^{i-1} a(n)\sqrt{n}(\bar{v}_i^n - w^n(i/n)).$$

Then for any $\delta > 0$

$$\lim_{n \rightarrow \infty} P \left[\max_{i \in \{1, \dots, n\}} \|W_i^n\| \geq \delta \right] = 0.$$

Proof. According to (2.14)

$$\frac{1}{n} \sum_{i=0}^{n-1} E[R(\eta_i^n \| \mu_{\bar{X}_i^n})] \leq \frac{K_E}{a^2(n)n}.$$

Because of this the (random) Radon-Nikodym derivatives

$$f_i^n(y) = \frac{d\eta_i^n}{d\mu_{\bar{X}_i^n}}(y)$$

are well defined and can be selected in a measurable way. We will control the magnitude of the noise when the Radon-Nikodym derivative is large by bounding

$$\frac{1}{n} \sum_{i=0}^{n-1} E[1_{\{f_i^n(\bar{v}_i^n) \geq r\}} \|\bar{v}_i^n\|]$$

for large r .

From the bound on the moment generating function (2.1),

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} e^{\frac{\lambda}{2d} \|y\|} \mu_x(dy) \leq 2^d e^{dK_{\text{mgf}}}.$$

Let $\sigma = \min\{\lambda/2^{d+1}, 1\}$ and recall the definition $\ell(b) \doteq b \log b - b + 1$. Then

$$\frac{1}{n} \sum_{i=0}^{n-1} E \left[1_{\{f_i^n(\bar{v}_i^n) \geq r\}} \|\bar{v}_i^n\| \right] = \frac{1}{n} \sum_{i=0}^{n-1} E \left[\int_{\{y: f_i^n(y) \geq r\}} \|y\| f_i^n(y) \mu_{\bar{X}_i^n}(dy) \right]$$

and the bound $ab \leq e^a + \ell(b)$ for $a, b \geq 0$ with $a = \sigma \|y\|$ and $b = f_i^n(y)$ gives that for all i

$$\begin{aligned} & E \left[\int_{\{y: f_i^n(y) \geq r\}} \|y\| f_i^n(y) \mu_{\bar{X}_i^n}(dy) \right] \\ & \leq \frac{1}{\sigma} E \left[\int_{\{y: f_i^n(y) \geq r\}} e^{\sigma \|y\|} \mu_{\bar{X}_i^n}(dy) \right] + \frac{1}{\sigma} E \left[\int_{\{y: f_i^n(y) \geq r\}} \ell(f_i^n(y)) \mu_{\bar{X}_i^n}(dy) \right]. \end{aligned}$$

Since $\ell(b) \geq 0$ for all $b \geq 0$

$$\begin{aligned} E \left[\int_{\{y: f_i^n(y) \geq r\}} \ell(f_i^n(y)) \mu_{\bar{X}_i^n}(dy) \right] & \leq E \left[\int_{\mathbb{R}^d} \ell(f_i^n(y)) \mu_{\bar{X}_i^n}(dy) \right] \\ & = E[R(\eta_i^n \| \mu_{\bar{X}_i^n})], \end{aligned}$$

and by Hölder's inequality

$$\begin{aligned} & E \left[\int_{\{y: f_i^n(y) \geq r\}} e^{\sigma \|y\|} \mu_{\bar{X}_i^n}(dy) \right] \\ & \leq E \left[\left(\int_{\mathbb{R}^d} 1_{\{f_i^n(y) \geq r\}} \mu_{\bar{X}_i^n}(dy) \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} e^{2\sigma \|y\|} \mu_{\bar{X}_i^n}(dy) \right)^{\frac{1}{2}} \right] \\ & = E \left[\mu_{\bar{X}_i^n}(\{y : f_i^n(y) \geq r\})^{\frac{1}{2}} \right] \left(2^d e^{dK_{\text{mgf}}} \right)^{\frac{1}{2}}. \end{aligned}$$

In addition Markov's inequality gives for $r \geq e^{-1}$

$$\mu_{\bar{X}_i^n}(\{y : f_i^n(y) \geq r\}) \leq \frac{1}{r \log r} \int \log(f_i^n(y)) f_i^n(y) \mu_{\bar{X}_i^n}(dy) = \frac{R(\eta_i^n \| \mu_{\bar{X}_i^n})}{r \log r}.$$

Therefore

$$\begin{aligned} & \frac{1}{n} \sum_{i=0}^{n-1} E \left[\int_{\{f_i^n(y) \geq r\}} \|y\| f_i^n(y) \mu_{\bar{X}_i^n}(dy) \right] \\ & \leq \frac{1}{\sigma} \left(2^d e^{dK_{\text{mgf}}} \right)^{\frac{1}{2}} \frac{1}{n} \sum_{i=0}^{n-1} E \left[\left(\frac{R(\eta_i^n \| \mu_{\bar{X}_i^n})}{r \log r} \right)^{\frac{1}{2}} \right] + \frac{1}{\sigma} \frac{1}{n} \sum_{i=0}^{n-1} E[R(\eta_i^n \| \mu_{\bar{X}_i^n})]. \end{aligned}$$

Since by Jensen's inequality

$$\frac{1}{n} \sum_{i=0}^{n-1} E \left[\left(\frac{R(\eta_i^n \| \mu_{\bar{X}_i^n})}{r \log r} \right)^{\frac{1}{2}} \right] \leq \left(\frac{1}{r \log r} \right)^{\frac{1}{2}} \left(\frac{1}{n} \sum_{i=0}^{n-1} E[R(\eta_i^n \| \mu_{\bar{X}_i^n})] \right)^{\frac{1}{2}},$$

we obtain the overall bound

$$\begin{aligned} & \frac{1}{n} \sum_{i=0}^{n-1} E \left[1_{\{f_i^n(\bar{v}_i^n) \geq r\}} \|\bar{v}_i^n\| \right] \\ & \leq \frac{1}{\sigma} \left(2^d e^{dK_{\text{mgf}}} \right)^{\frac{1}{2}} \left(\frac{1}{r \log r} \right)^{\frac{1}{2}} \left(\frac{1}{n} \sum_{i=0}^{n-1} E[R(\eta_i^n \| \mu_{\bar{X}_i^n})] \right)^{\frac{1}{2}} \\ & \quad + \frac{1}{\sigma} \frac{1}{n} \sum_{i=0}^{n-1} E[R(\eta_i^n \| \mu_{\bar{X}_i^n})] \\ & \leq \frac{1}{\sigma} \frac{K_E^{\frac{1}{2}}}{a(n)\sqrt{n}} \left(2^d e^{dK_{\text{mgf}}} \right)^{\frac{1}{2}} \left(\frac{1}{r \log r} \right)^{\frac{1}{2}} + \frac{1}{\sigma} \frac{K_E}{a(n)^2 n}. \end{aligned} \tag{3.7}$$

Using this result we can complete the proof. Define

$$\xi_i^{n,r} \doteq \begin{cases} \bar{v}_i^n & \text{if } f_i^n(\bar{v}_i^n) < r \\ 0 & \text{otherwise.} \end{cases}$$

For any for any $\delta > 0$

$$\begin{aligned}
& P \left\{ \max_{k=0, \dots, n-1} \left\| \frac{1}{n} \sum_{i=0}^k a(n) \sqrt{n} \left(\bar{v}_i^n - w^n \left(\frac{i}{n} \right) \right) \right\| \geq 3\delta \right\} \\
& \leq P \left\{ \max_{k=0, \dots, n-1} \left\| \frac{1}{n} \sum_{i=0}^k a(n) \sqrt{n} (\bar{v}_i^n - \xi_i^{n,r}) \right\| \geq \delta \right\} \\
& \quad + P \left\{ \max_{k=0, \dots, n-1} \left\| \frac{1}{n} \sum_{i=0}^k a(n) \sqrt{n} \left(\xi_i^{n,r} - \int_{\{y: f_i^n(y) < r\}} y \eta_i^n(dy) \right) \right\| \geq \delta \right\} \\
& \quad + P \left\{ \max_{k=0, \dots, n-1} \left\| \frac{1}{n} \sum_{i=0}^k a(n) \sqrt{n} \left(w^n \left(\frac{i}{n} \right) - \int_{\{y: f_i^n(y) < r\}} y \eta_i^n(dy) \right) \right\| \geq \delta \right\}.
\end{aligned}$$

The first term satisfies

$$\begin{aligned}
& P \left\{ \max_{k=0, \dots, n-1} \left\| \frac{1}{n} \sum_{i=0}^k a(n) \sqrt{n} (\bar{v}_i^n - \xi_i^{n,r}) \right\| \geq \delta \right\} \\
& \leq \frac{1}{\delta} a(n) \sqrt{n} \frac{1}{n} \sum_{i=0}^{n-1} E [\|\bar{v}_i^n - \xi_i^{n,r}\|] \\
& = \frac{1}{\delta} a(n) \sqrt{n} \frac{1}{n} \sum_{i=0}^{n-1} E \left[1_{\{f_i^n(\bar{v}_i^n) \geq r\}} \|\bar{v}_i^n\| \right].
\end{aligned}$$

The second term is a submartingale so by Doob's submartingale inequality

$$\begin{aligned}
& P \left\{ \max_{k=0, \dots, n-1} \left\| \frac{1}{n} \sum_{i=0}^k a(n) \sqrt{n} \left(\xi_i^{n,r} - \int_{\{y: f_i^n(y) < r\}} y \eta_i^n(dy) \right) \right\| \geq \delta \right\} \\
& \leq \frac{1}{\delta^2} E \left[\left\| \frac{1}{n} \sum_{i=0}^{n-1} a(n) \sqrt{n} \left(\xi_i^{n,r} - \int_{\{y: f_i^n(y) < r\}} y \eta_i^n(dy) \right) \right\|^2 \right] \\
& = \frac{1}{\delta^2} \frac{a(n)^2}{n} \sum_{i=0}^{n-1} E \left[\left\| \left(\xi_i^{n,k} - \int_{\{y: f_i^n(y) < r\}} y \eta_i^n(dy) \right) \right\|^2 \right] \\
& \leq \frac{1}{\delta^2} \frac{a(n)^2}{n} \sum_{i=0}^{n-1} E \left[\left\| \xi_i^{n,k} \right\|^2 \right] \\
& = \frac{1}{\delta^2} \frac{a(n)^2}{n} \sum_{i=0}^{n-1} E \left[\int_{\{y: f_i^n(y) < r\}} \|y\|^2 f_i^n(y) \mu_{\bar{X}_i^n}(dy) \right] \\
& \leq \frac{r}{\delta^2} \frac{a(n)^2}{n} \sum_{i=0}^{n-1} E \left[\int_{\mathbb{R}^d} \|y\|^2 \mu_{\bar{X}_i^n}(dy) \right] \\
& \leq \frac{r}{\delta^2} a(n)^2 K_{\mu,2},
\end{aligned}$$

where

$$K_{\mu,2} = \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \|y\|^2 \mu_x(dy) < \infty,$$

and the finiteness is due to (2.1). We can use Jensen's inequality with the third term and get the same bound that was shown for the first. We have

$$\begin{aligned}
& P \left\{ \max_{k=0, \dots, n-1} \left\| \frac{1}{n} \sum_{i=0}^k a(n) \sqrt{n} \left(w^n \left(\frac{i}{n} \right) - \int_{\{y: f_i^n(y) < r\}} y \eta_i^n(dy) \right) \right\| \geq \delta \right\} \\
& \leq \frac{1}{\delta} a(n) \sqrt{n} \frac{1}{n} \sum_{i=0}^{n-1} E \left[\left\| \left(w^n \left(\frac{i}{n} \right) - \int_{\{y: f_i^n(y) < r\}} y \eta_i^n(dy) \right) \right\| \right] \\
& = \frac{1}{\delta} a(n) \sqrt{n} \frac{1}{n} \sum_{i=0}^{n-1} E \left[\left\| \int_{\{y: f_i^n(y) \geq r\}} y \eta_i^n(dy) \right\| \right] \\
& \leq \frac{1}{\delta} a(n) \sqrt{n} \frac{1}{n} \sum_{i=0}^{n-1} E \left[\int_{\{y: f_i^n(y) \geq r\}} \|y\| \eta_i^n(dy) \right] \\
& = \frac{1}{\delta} a(n) \sqrt{n} \frac{1}{n} \sum_{i=0}^{n-1} E \left[1_{\{f_i^n(\bar{v}_i^n) \geq r\}} \|\bar{v}_i^n\| \right].
\end{aligned}$$

Combining the bounds for these three terms with (3.7) gives

$$\begin{aligned}
& P \left\{ \max_{k=0, \dots, n-1} \left\| \frac{1}{n} \sum_{i=0}^k a(n) \sqrt{n} \left(\bar{v}_i^n - w^n \left(\frac{i}{n} \right) \right) \right\| \geq 3\delta \right\} \\
& \leq \frac{2}{\delta} a(n) \sqrt{n} \frac{1}{n} \sum_{i=0}^{n-1} E \left[1_{\{f_i^n(\bar{v}_i^n) \geq r\}} \|\bar{v}_i^n\| \right] + \frac{r}{\delta^2} a(n)^2 K_{\mu, 2} \\
& \leq \frac{2}{\sigma \delta} K_E^{\frac{1}{2}} \left(2^d e^{dK_{\text{mgf}}} \right)^{\frac{1}{2}} \left(\frac{1}{r \log r} \right)^{\frac{1}{2}} + \frac{2}{\sigma \delta} \frac{K_E}{a(n) \sqrt{n}} + a(n)^2 \frac{r}{\delta^2} K_{\mu, 2}.
\end{aligned}$$

Choosing $r = 1/a(n)$ and using $a(n) \rightarrow 0, a(n) \sqrt{n} \rightarrow \infty$ gives

$$P \left\{ \max_{k=0, \dots, n-1} \left\| \frac{1}{n} \sum_{i=0}^k a(n) \sqrt{n} \left(\bar{v}_i^n - w^n \left(\frac{i}{n} \right) \right) \right\| \geq 3\delta \right\} \rightarrow 0$$

as $n \rightarrow \infty$, which completes the proof. ■

This theorem, combined with the following discrete version of Gronwall's inequality, will allow us to prove $\bar{Y}^n - \check{Y}^n \rightarrow 0$.

Lemma 3.7 *If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are nonnegative sequences defined for $n = 0, 1, \dots$ and satisfying*

$$a_n \leq c_n + \sum_{k=0}^{n-1} b_k a_k,$$

then

$$a_n \leq c_n + \sum_{k=0}^{n-1} b_k c_k \exp \left\{ \sum_{i=k+1}^{n-1} b_i \right\}.$$

Theorem 3.8 *Under the conditions of Theorem 3.6 $\check{Y}^n - \bar{Y}^n \rightarrow 0$ in probability.*

Proof. Recall that

$$\bar{Y}_k^n = \sum_{i=0}^{k-1} \frac{a(n)}{\sqrt{n}} \left(b \left(X_i^{n,0} + \frac{1}{a(n)\sqrt{n}} \bar{Y}_i^n \right) - b \left(X_i^{n,0} \right) \right) + \sum_{i=0}^{k-1} \frac{a(n)}{\sqrt{n}} \bar{v}_i^n$$

and

$$\check{Y}_k^n = \sum_{i=0}^{k-1} \frac{a(n)}{\sqrt{n}} \left(b \left(X_i^{n,0} + \frac{1}{a(n)\sqrt{n}} \check{Y}_i^n \right) - b \left(X_i^{n,0} \right) \right) + \sum_{i=0}^{k-1} \frac{a(n)}{\sqrt{n}} w^n \left(\frac{i}{n} \right),$$

so with W_k^n defined as in Theorem 3.6

$$\|\bar{Y}_k^n - \check{Y}_k^n\| \leq \|W_k^n\| + \sum_{i=0}^{k-1} \frac{K_b}{n} \|\bar{Y}_i^n - \check{Y}_i^n\|.$$

Using Lemma 3.7 gives

$$\begin{aligned} \|\bar{Y}_k^n - \check{Y}_k^n\| &\leq \|W_k^n\| + \sum_{i=0}^{k-1} \|W_i^n\| \frac{K_b}{n} \exp \left\{ \frac{K_b}{n} (k - i - 1) \right\} \\ &\leq (1 + K_b e^{K_b}) \max_{i \in \{1, \dots, k\}} \{\|W_i^n\|\} \end{aligned}$$

so

$$\max_{i \in \{1, \dots, n\}} \{\|\bar{Y}_i^n - \check{Y}_i^n\|\} \leq (1 + K_b e^{K_b}) \max_{i \in \{1, \dots, n\}} \{\|W_i^n\|\}.$$

Since $\max_{i \in \{1, \dots, n\}} \{\|W_i^n\|\} \rightarrow 0$ in probability

$$\max_{i \in \{1, \dots, n\}} \{\|\bar{Y}_i^n - \check{Y}_i^n\|\} \rightarrow 0 \text{ and hence } \sup_{t \in [0, 1]} \|\bar{Y}^n(t) - \check{Y}^n(t)\| \rightarrow 0$$

in probability. ■

To complete the proof of the convergence we need to show $\check{Y}^n - \hat{Y}^n \rightarrow 0$. Recall that these two processes have the same driving terms but different drifts, in that \hat{Y}^n satisfies the ODE

$$\hat{Y}^n(t) = \int_0^t Db(X^0(s)) \hat{Y}^n(s) ds + \int_0^t \hat{w}^n(s) ds$$

while \check{Y}^n is the linear interpolation of the discrete time process defined by $\check{Y}_0^n = 0$ and

$$\check{Y}_{i+1}^n = \check{Y}_i^n + \frac{a(n)}{\sqrt{n}} \left(b \left(X_i^{n,0} + \frac{1}{a(n)\sqrt{n}} \check{Y}_i^n \right) - b \left(X_i^{n,0} \right) \right) + \frac{1}{n} \hat{w}^n \left(\frac{i}{n} \right).$$

However, essentially the same arguments as those used in Lemma 3.4 to show tightness of $\{\hat{Y}^n\}$ can be used to prove tightness of $\{\check{Y}^n\}$, and then it easily follows as in Lemma 3.5 that any limit will satisfy the same ODE (2.15) as the limit of $\{\hat{Y}^n\}$, and therefore $\check{Y}^n - \hat{Y}^n \rightarrow 0$ follows.

Combining $\bar{Y}^n - \check{Y}^n \rightarrow 0$, $\check{Y}^n - \hat{Y}^n \rightarrow 0$, and $\hat{Y}^n \rightarrow \hat{Y}$ demonstrates that along the subsequence where $\hat{\eta}^n \rightarrow \hat{\eta}$ weakly $\bar{Y}^n \rightarrow \hat{Y}$ in distribution, which implies that along this subsequence $(\hat{\eta}^n, \bar{Y}^n) \rightarrow (\hat{\eta}, \hat{Y})$ weakly. We have already shown that with probability 1 $\hat{\eta}_2(dt)$ is Lebesgue measure and

$$\hat{Y}(t) = \int_0^t Db(X^0(s))\hat{Y}(s)ds + \int_0^t \int_{\mathbb{R}^d} y\hat{\eta}_{1|2}(dy|t)ds,$$

so the proof of convergence (i.e., the first part of Theorem 2.5) is complete.

To finish Theorem 2.5 we must lastly show the bound (2.16). Note that the weak convergence of \bar{Y}^n implies

$$\sup_{t \in [0,1]} \|\bar{X}^n(\lfloor nt \rfloor / n) - X^0(t)\| \rightarrow 0 \text{ in probability.} \quad (3.8)$$

Now define random measures on $\mathbb{R}^d \otimes \mathbb{R}^d \otimes [0, 1]$ by

$$\gamma^n(dx \otimes dy \otimes dt) = \delta_{\bar{X}^n(\lfloor nt \rfloor / n)}(dx) \hat{\eta}^n(dy \otimes dt).$$

Note that the tightness of $\{\gamma^n\}$ follows easily from (3.8) and from the tightness of $\{\hat{\eta}^n\}$. Thus given any subsequence we can choose a further subsequence (again we will retain n as the index for simplicity) along which $\{\gamma^n\}$ converges weakly to some limit γ on $\mathcal{P}(\mathbb{R}^d \otimes \mathbb{R}^d \otimes [0, 1])$ with

$$\gamma_{2,3}(dy \otimes dt) = \hat{\eta}(dy \otimes dt),$$

where $\gamma_{2,3}$ is the second and third marginal of γ . If we establish (2.16) for this subsequence it follows for the original one using a standard argument by contradiction. For $\sigma > 0$ let

$$G_\sigma^{X^0} = \{(x, y, t) : \|x - X^0(t)\| \leq \sigma\}$$

be closed sets centered around $X^0(t)$ in the x variable, and note that by (3.8) and weak convergence, for all $\sigma > 0$

$$1 = \limsup_{n \rightarrow \infty} E \left[\gamma^n \left(G_\sigma^{X^0} \right) \right] \leq E \left[\gamma \left(G_\sigma^{X^0} \right) \right].$$

Thus

$$E \left[\gamma \left(\cap_{n \in \mathbb{N}} G_{1/n}^{X^0} \right) \right] = 1,$$

so with probability 1 γ puts all its mass on $\{(x, y, t) : x = X^0(t)\}$. Therefore with probability 1, for a.e. (y, t) under $\gamma_{2,3}(dy \otimes dt)$,

$$\gamma_{1|2,3}(dx|y, t) = \delta_{X^0(t)}(dx).$$

Combined with the fact that the second marginal of $\hat{\eta}(dy \otimes dt)$ is Lebesgue measure, this gives

$$\gamma(dx \otimes dy \otimes dt) = \delta_{X^0(t)}(dx) \hat{\eta}(dy|t) dt. \quad (3.9)$$

Let

$$\bar{L}_K(x, \beta) = \sup_{\alpha \in \mathbb{R}^d} \left\{ \langle \alpha, \beta \rangle - \frac{1}{2} \|\alpha\|_{A(x)}^2 - \frac{1}{2K} \|\alpha\|^2 \right\}.$$

Then uniformly in x and compact subsets of β

$$\liminf_{n \rightarrow \infty} a(n)^2 n L_c \left(x, \frac{1}{a(n)\sqrt{n}} \beta \right) \geq \bar{L}_K(x, \beta),$$

and as $K \rightarrow \infty$

$$\bar{L}_K(x, \beta) \uparrow \frac{1}{2} \|\beta\|_{A^{-1}(x)}^2$$

for all $(x, \beta) \in \mathbb{R}^{2d}$. Combining this with Lemma 3.1 and using Fatou's lemma for weak convergence,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} a(n)^2 n E \left[\sum_{i=0}^{n-1} \frac{1}{n} R(\eta_i^n \| \mu_{\bar{X}_i^n}) \right] \\ & \geq \liminf_{n \rightarrow \infty} E \left[\int_{\mathbb{R}^d \otimes \mathbb{R}^d \otimes [0,1]} a(n)^2 n L_c \left(x, \frac{1}{a(n)\sqrt{n}} y \right) \gamma^n(dx \otimes dy \otimes dt) \right] \\ & \geq E \left[\int_{\mathbb{R}^d \otimes \mathbb{R}^d \otimes [0,1]} \bar{L}_K(x, y) \gamma(dx \otimes dy \otimes dt) \right] \end{aligned}$$

for all K . Then using the monotone convergence theorem, the decomposition (3.9), and Jensen's inequality in that order shows that

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} a(n)^2 n E \left[\sum_{i=0}^{n-1} \frac{1}{n} R(\eta_i^n \| \mu_{\bar{X}_i^n}) \right] \\
& \geq \lim_{K \rightarrow \infty} E \left[\int_{\mathbb{R}^d \otimes \mathbb{R}^d \otimes [0,1]} \bar{L}_K(x, y) \gamma(dx \otimes dy \otimes dt) \right] \\
& = E \left[\int_{\mathbb{R}^d \otimes \mathbb{R}^d \otimes [0,1]} \frac{1}{2} \|y\|_{A^{-1}(x)}^2 \gamma(dx \otimes dy \otimes dt) \right] \\
& = E \left[\int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} \|y\|_{A^{-1}(X^0(t))}^2 \hat{\eta}(dy|t) dt \right] \\
& \geq E \left[\frac{1}{2} \int_0^1 \|\hat{w}(t)\|_{A^{-1}(X^0(t))}^2 dt \right],
\end{aligned}$$

which is (2.16).

4 Laplace Upper Bound

The goal of this section is to prove (2.12), which due to the minus sign corresponds to the Laplace upper bound. Suppose for each n that η^n comes within ε of achieving the infimum in (2.9), so that

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} -a(n)^2 \log E \left[e^{-\frac{1}{a(n)^2} F(Y^n)} \right] + \varepsilon \\
& \geq \liminf_{n \rightarrow \infty} E \left[\sum_{i=0}^{n-1} a(n)^2 R(\eta_i^n \| \mu_{\bar{X}_i^n}) + F(\bar{Y}^n) \right]. \tag{4.1}
\end{aligned}$$

Since $\sup_{x \in \mathbb{R}^d} |F(x)| \leq K_F$ for some $K_F < \infty$, we also have

$$\sup_n a(n)^2 n E \left[\sum_{i=0}^{n-1} \frac{1}{n} R(\eta_i^n \| \mu_{\bar{X}_i^n}) \right] \leq 2K_F + \varepsilon.$$

Consequently we can choose a subsequence of $\{\eta^n\}$ (we retain n as the index for convenience) along which the conclusions of Theorem 2.5 hold. Combin-

ing this with (4.1) gives

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} -a(n)^2 \log E \left[e^{-\frac{1}{a(n)^2} F(Y^n)} \right] + \varepsilon \\
& \geq \liminf_{n \rightarrow \infty} E \left[\sum_{i=0}^{n-1} a(n)^2 R(\eta_i^n \| \mu_{\bar{X}_i^n}) + F(\bar{Y}^n) \right] \\
& \geq E \left[\int_0^1 \frac{1}{2} \|\hat{w}(s)\|_{A^{-1}(X^0(s))}^2 ds + F(\hat{Y}) \right].
\end{aligned}$$

Recalling

$$\hat{Y}(t) = \int_0^t Db(X^0(s)) \hat{Y}(s) ds + \int_0^t \hat{w}(s) ds,$$

it follows that

$$\begin{aligned}
& E \left[\int_0^1 \frac{1}{2} \|\hat{w}(s)\|_{A^{-1}(X^0(s))}^2 ds + F(\hat{Y}) \right] \\
& \geq \inf_{u \in L^2([0,1]; \mathbb{R}^d)} \left\{ \int_0^1 \frac{1}{2} \|u(s)\|_{A^{-1}(X^0(s))}^2 ds + F(\phi^u) \right\} \\
& = \inf_{u \in L^2([0,1]; \mathbb{R}^d)} \left\{ \int_0^1 \frac{1}{2} \|u(s)\|^2 ds + F(\phi^{A^{1/2}(X^0)u}) \right\},
\end{aligned}$$

with ϕ^u defined as in (2.8). Since $\varepsilon > 0$ is arbitrary, we have the lower bound (2.12).

5 Laplace Lower Bound

The goal of this section is to prove (2.13). Note that for $u, v \in L^2([0,1] : \mathbb{R}^d)$

$$\begin{aligned}
& \phi^{A^{1/2}(X^0)u}(t) - \phi^{A^{1/2}(X^0)v}(t) \\
& = \int_0^t Db(X^0(s)) \left(\phi^{A^{1/2}(X^0)u}(s) - \phi^{A^{1/2}(X^0)v}(s) \right) ds \\
& \quad + \int_0^t A^{1/2}(X^0(s))(u(s) - v(s)) ds.
\end{aligned}$$

Thus by Gronwall's inequality

$$\begin{aligned} & \sup_{t \in [0,1]} \left\| \phi^{A^{1/2}(X^0)u}(t) - \phi^{A^{1/2}(X^0)v}(t) \right\| \\ & \leq (1 + K_b e^{K_b}) K_A^{1/2} \int_0^1 \|u(s) - v(s)\| ds \\ & \leq (1 + K_b e^{K_b}) K_A^{1/2} \left(\int_0^1 \|u(s) - v(s)\|^2 ds \right)^{\frac{1}{2}}. \end{aligned} \quad (5.1)$$

Since $C([0,1] : \mathbb{R}^d)$ is dense in $L^2([0,1] : \mathbb{R}^d)$, the proof of the Laplace lower bound is reduced to showing that for an arbitrary $u \in C([0,1] : \mathbb{R}^d)$

$$\limsup_{n \rightarrow \infty} -a(n)^2 \log E \left[e^{-\frac{1}{a(n)^2} F(Y^n)} \right] \leq \frac{1}{2} \int_0^1 \|u(s)\|^2 ds + F \left(\phi^{A^{1/2}(X^0)u} \right). \quad (5.2)$$

The main difficulty is to deal with the possible degeneracy of the noise. Recall the orthogonal decomposition of $A^{-1}(x)$ (2.2). Define

$$A_K^{-1}(x) = Q(x) \Lambda_K^{-1}(x) Q^T(x)$$

where $\Lambda_K^{-1}(x)$ is the diagonal matrix such that $\Lambda_{ii,K}^{-1}(x) = \Lambda_{ii}^{-1}(x)$ when $\Lambda_{ii}^{-1}(x) \leq K^2$ and $\Lambda_{ii,K}^{-1}(x) = K^2$ when $\Lambda_{ii}^{-1}(x) > K^2$. Note that by [18, Theorem 6.2.37] $A^{1/2}(x)$, $A_K^{-1}(x)$ and $A_K^{1/2}(x)$ are continuous functions of $A(x)$, and consequently they are also continuous functions of $x \in \mathbb{R}^d$. In addition define

$$u_K(s) = \begin{cases} u(s) & \text{for } \|u(s)\| \leq K \\ \frac{K u(s)}{\|u(s)\|} & \text{for } \|u(s)\| > K \end{cases}.$$

Let $\phi^{u,K}(t) = \phi^{A(X^0)A_K^{-1/2}(X^0)u_K}(t)$, and note that $\phi^{u,K}$ solves

$$\begin{aligned} \phi^{u,K}(t) &= \int_0^t Db(X^0(s)) \phi^{u,K}(s) ds \\ &+ \int_0^t A(X^0(s)) A_K^{-1/2}(X^0(s)) u_K(s) ds. \end{aligned} \quad (5.3)$$

To simplify notation we define $s_i^n \doteq i/n$ and $s^n(t) = \lfloor nt \rfloor / n$, where $\lfloor a \rfloor$ is the integer part of a . Note that $s^n(t) - t \rightarrow 0$ uniformly for $t \in [0,1]$ as $n \rightarrow \infty$. For n sufficiently large

$$\max_{0 \leq i \leq n-1} \left\{ \frac{1}{a(n)\sqrt{n}} \left\| A_K^{-1/2}(X^0(s_i^n)) u_K(s_i^n) \right\| \right\} \leq \frac{1}{a(n)\sqrt{n}} K^2 \leq \lambda_{DA}$$

and we can define the sequence $\{(\bar{X}^{n,u,K}, \bar{Y}^{n,u,K}, \eta^{n,u,K}, \hat{\eta}^{n,u,K})\}$ as in Construction 2.4 with

$$\begin{aligned} \eta_i^{n,u,K}(dy) &= \exp \left\{ \left\langle y, \frac{1}{a(n)\sqrt{n}} A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n) \right\rangle \right. \\ &\quad \left. - H_c \left(\bar{X}_i^{n,u,K}, \frac{1}{a(n)\sqrt{n}} A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n) \right) \right\} \mu_{\bar{X}_i^{n,u,K}}(dy). \end{aligned}$$

Using (2.3) and the fact that

$$\int_{\mathbb{R}^d} y \exp\{\langle y, \alpha \rangle - H_c(x, \alpha)\} \mu_x(dy) = D_\alpha H_c(x, \alpha)$$

we have for $\|\alpha\| \leq \lambda_{DA}$

$$\left\| \int_{\mathbb{R}^d} y \exp\{\langle y, \alpha \rangle - H_c(x, \alpha)\} \mu_x(dy) - A(x)\alpha \right\| \leq K_{DA} \|\alpha\|^2. \quad (5.4)$$

The next result identifies the limit in probability of the controlled processes and an asymptotic bound for the relative entropies.

Theorem 5.1 *Let $u \in C([0, 1] : \mathbb{R}^d)$ and $K < \infty$ be given, construct $\{(\bar{X}^{n,u,K}, \bar{Y}^{n,u,K}, \eta^{n,u,K}, \hat{\eta}^{n,u,K})\}$ as in this section and define $\phi^{u,K}$ by (5.3). Then*

$$\bar{Y}^{n,u,K} \rightarrow \phi^{u,K} \quad (5.5)$$

in $C([0, 1] : \mathbb{R}^d)$ in probability, and

$$\begin{aligned} \limsup_{n \rightarrow \infty} a^2(n) n E \left[\frac{1}{n} \sum_{i=0}^{n-1} R \left(\eta_i^{n,u,K} \parallel \mu_{\bar{X}_i^{n,u,K}} \right) \right] \\ \leq \frac{1}{2} \int_0^1 \left\| A_K^{-1/2} (X^0(s)) u_K(s) \right\|_{A(X^0(s))}^2 ds. \end{aligned} \quad (5.6)$$

Proof. From (2.4) and (5.4) we have for all n sufficiently large that $\frac{1}{a(n)\sqrt{n}} K^2 \leq$

λ_{DA}

$$\begin{aligned}
& R \left(\eta_i^{n,u,K} \left\| \mu_{\bar{X}_i^{n,u,K}} \right\| \right) \\
&= \int_{\mathbb{R}^d} \left\langle y, \frac{1}{a(n)\sqrt{n}} A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n) \right\rangle \eta_i^{n,u,K}(dy) \\
&\quad - H_c \left(\bar{X}_i^{n,u,K}, \frac{1}{a(n)\sqrt{n}} A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n) \right) \\
&\leq \left\langle \frac{1}{a(n)\sqrt{n}} A \left(\bar{X}_i^{n,u,K} \right) A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n), \right. \\
&\quad \left. \frac{1}{a(n)\sqrt{n}} A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n) \right\rangle \\
&\quad - \frac{1}{2} \left\langle \frac{1}{a(n)\sqrt{n}} A \left(\bar{X}_i^{n,u,K} \right) A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n), \right. \\
&\quad \left. \frac{1}{a(n)\sqrt{n}} A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n) \right\rangle + \frac{2}{a(n)^3 n^{3/2}} K_{DA} K^6 \\
&= \frac{1}{2a(n)^2 n} \left\| A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n) \right\|_{A(\bar{X}_i^{n,u,K})}^2 + \frac{2}{a(n)^3 n^{3/2}} K_{DA} K^6.
\end{aligned}$$

Consequently

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} a^2(n) n E \left[\frac{1}{n} \sum_{i=0}^{n-1} R \left(\eta_i^{n,u,K} \left\| \mu_{\bar{X}_i^{n,u,K}} \right\| \right) \right] \\
& \leq \limsup_{n \rightarrow \infty} \frac{1}{2} E \left[\frac{1}{n} \sum_{i=0}^{n-1} \left\| A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n) \right\|_{A(\bar{X}_i^{n,u,K})}^2 \right],
\end{aligned} \tag{5.7}$$

where in fact

$$\limsup_{n \rightarrow \infty} \frac{1}{2} E \left[\frac{1}{n} \sum_{i=0}^{n-1} \left\| A_K^{-1/2} (X^0(s_i^n)) u_K(s_i^n) \right\|_{A(\bar{X}_i^{n,u,K})}^2 \right] \leq \frac{1}{2} K^4 K_A.$$

Therefore (2.14) is satisfied by $\{\eta^{n,u,K}\}$, so we can apply Theorem 2.5 and choose a subsequence (keeping n as the index for convenience) along which $\{(\hat{\eta}^{n,u,K}, \bar{Y}^{n,u,K})\}$ converges weakly to some limit $(\hat{\eta}^{u,K}, \hat{Y}^{u,K})$, where $\hat{\eta}_2^{u,K}$ is Lebesgue measure and

$$\hat{Y}^{u,K}(t) = \int_0^t Db(X^0(s)) \hat{Y}^{u,K}(s) ds + \int_0^t \int_{\mathbb{R}^d} y \hat{\eta}_{1|2}^{u,K}(dy|s) ds.$$

This implies

$$\sup_{t \in [0,1]} \|\bar{X}^{n,u,K}(t) - X^0(t)\| \rightarrow 0 \quad (5.8)$$

in probability. Because of this, the uniform bound on $A^{1/2}(x)$ and the continuity of $A^{1/2}(x)$, we have (recall that $s^n(t) \doteq \lfloor nt \rfloor / n$)

$$\sup_{t \in [0,1]} \left\| A^{1/2}(\bar{X}^{n,u,K}(s^n(t))) - A^{1/2}(X^0(s^n(t))) \right\| \rightarrow 0$$

in probability. However, the continuity of $A^{1/2}(X^0)A_K^{-1/2}(X^0)u_K$ gives

$$\begin{aligned} \sup_{t \in [0,1]} \left\| A^{1/2}(X^0(s^n(t)))A_K^{-1/2}(X^0(s^n(t)))u_K(s^n(t)) \right. \\ \left. - A^{1/2}(X^0(t))A_K^{-1/2}(X^0(t))u_K(t) \right\| \rightarrow 0. \end{aligned}$$

Combining these limits, and using the fact that $A_K^{-1/2}(X^0)u_K$ is uniformly bounded, shows that

$$\begin{aligned} \sup_{t \in [0,1]} \left\| A^{1/2}(\bar{X}^{n,u,K}(s^n(t)))A_K^{-1/2}(X^0(s^n(t)))u_K(s^n(t)) \right. \\ \left. - A^{1/2}(X^0(t))A_K^{-1/2}(X^0(t))u_K(t) \right\| \rightarrow 0 \end{aligned} \quad (5.9)$$

in probability. This combined with the uniform bounds allows us to use dominated convergence to get

$$\begin{aligned} \limsup_{n \rightarrow \infty} E \left[\frac{1}{2} \int_0^1 \left\| A_K^{-1/2}(X^0(s^n(t)))u_K(s^n(t)) \right\|_{A(\bar{X}^{n,u,K}(s^n(t)))}^2 dt \right] \\ = \frac{1}{2} \int_0^1 \left\| A_K^{-1/2}(X^0(t))u_K(t) \right\|_{A(X^0(t))}^2 dt. \end{aligned}$$

Combining this with (5.7) shows (5.6).

To prove (5.5) we will show that in fact

$$\hat{\eta}^{u,K}(dy \otimes dt) = \delta_{A(X^0(t))A_K^{-1/2}(X^0(t))u_K(t)}(dy)dt.$$

For all $\sigma > 0$ let

$$G_\sigma = \left\{ (z, t) \in \mathbb{R}^d \times [0, 1] : \left\| z - A(X^0(t))A_K^{-1/2}(X^0(t))u_K(t) \right\| \leq \sigma \right\},$$

and note that by weak convergence $\limsup_{n \rightarrow \infty} E[\hat{\eta}^{n,u,K}(G_\sigma)] \leq E[\hat{\eta}^{u,K}(G_\sigma)]$. Note also that

$$\begin{aligned} & E[\hat{\eta}^{n,u,K}(G_\sigma)] \\ & \geq P \left[\sup_{t \in [0,1]} \left\| a(n)\sqrt{n} \int_{\mathbb{R}^d} y \eta_{[nt]}^{n,u,K}(dy) - A(X^0(t)) A_K^{-1/2}(X^0(t)) u_K(t) \right\| \leq \sigma \right]. \end{aligned}$$

However, by (5.4) we can choose n large enough to make

$$\begin{aligned} & \sup_{t \in [0,1]} \left\| a(n)\sqrt{n} \int_{\mathbb{R}^d} y \eta_{[nt]}^{n,u,K}(dy) \right. \\ & \quad \left. - A(\bar{X}^{n,u,K}(s^n(t))) A_K^{-1/2}(X^0(s^n(t))) u_K(s^n(t)) \right\| \end{aligned}$$

arbitrarily small, and the proof that

$$\begin{aligned} & \sup_{t \in [0,1]} \left\| A(\bar{X}^{n,u,K}(s^n(t))) A_K^{-1/2}(X^0(s^n(t))) u_K(s^n(t)) \right. \\ & \quad \left. - A(X^0(t)) A_K^{-1/2}(X^0(t)) u_K(t) \right\| \rightarrow 0 \end{aligned}$$

in probability is identical to the proof of (5.9). Therefore $\limsup_{n \rightarrow \infty} E[\hat{\eta}^{u,K,n}(G_\sigma)] = 1$ for all $\sigma > 0$, and so $E[\hat{\eta}^{u,K}(\cap_{n \in \mathbb{N}} G_{1/n})] = 1$. This implies that with probability 1

$$\hat{\eta}_{1|2}^{u,K}(dy|t) = \delta_{A(X^0(t)) A_K^{-1/2}(X^0(t)) u_K(t)}(dy)$$

for a.e. t . It follows that

$$\hat{Y}^{u,K}(t) = \int_0^t Db(X^0(s)) \hat{Y}^{u,K}(s) ds + \int_0^t A(X^0(s)) A_K^{-1/2}(X^0(s)) u_K(s) ds,$$

and therefore $\bar{Y}^{n,u,K} \rightarrow \phi^{u,K}$ weakly. This implies (5.5) and completes the proof. ■

The second theorem in this section allows us to approximate $F(\phi^{A^{1/2}(X^0)u})$ by $F(\phi^{u,K})$ and $\frac{1}{2} \int_0^1 \|u(s)\|^2 ds$ by

$$\frac{1}{2} \int_0^1 \left\| A_K^{-1/2}(X^0(s)) u_K(s) \right\|_{A(X^0(s))}^2 ds.$$

Theorem 5.2 *Let $u \in C([0,1] : \mathbb{R}^d)$ and define $\phi^{A^{1/2}(X^0)u}$ by (2.8) and $\phi^{u,K}$ by (5.3). Then*

$$\phi^{u,K} \rightarrow \phi^{A^{1/2}(X^0)u}$$

in $C([0, 1] : \mathbb{R}^d)$ and

$$\frac{1}{2} \int_0^1 \left\| A_K^{-1/2}(X^0(s)) u_K(s) \right\|_{A(X^0(s))}^2 ds \rightarrow \frac{1}{2} \int_0^1 \|u(s)\|^2 ds.$$

Proof. Note that

$$\left\| A^{1/2}(X^0(s)) A_K^{-1/2}(X^0(s)) u_K(s) \right\| \leq \|u(s)\|$$

for all $s \in [0, 1]$ and

$$A^{1/2}(X^0(s)) A_K^{-1/2}(X^0(s)) u_K(s) \rightarrow u(s) \quad (5.10)$$

pointwise. Since $u \in L^2([0, 1] : \mathbb{R}^d)$, by dominated convergence (5.10) also holds in $L^2([0, 1] : \mathbb{R}^d)$. This gives

$$\frac{1}{2} \int_0^1 \left\| A_K^{-1/2}(X^0(s)) u_K(s) \right\|_{A(X^0(s))}^2 ds \rightarrow \frac{1}{2} \int_0^1 \|u(s)\|^2 ds.$$

Combining this with (5.1) shows that

$$\phi^{u,K} \rightarrow \phi^{A^{1/2}(X^0)u}$$

in $C([0, 1] : \mathbb{R}^d)$. ■

Using (2.9) and the fact that any given control is suboptimal,

$$\begin{aligned} & -a(n)^2 \log E \left[e^{-\frac{1}{a(n)^2} F(Y^n)} \right] \\ & \leq E \left[\sum_{i=0}^{n-1} a(n)^2 R \left(\eta_i^{n,u,K} \left\| \mu_{\bar{X}_i^{n,u,K}} \right\| + F(\bar{Y}^{n,u,K}) \right) \right]. \end{aligned}$$

Using Theorem 5.1, this implies

$$\begin{aligned} & \limsup_{n \rightarrow \infty} -a(n)^2 \log E \left[e^{-\frac{1}{a(n)^2} F(Y^n)} \right] \\ & \leq \frac{1}{2} \int_0^1 \left\| A_K^{-1/2}(X^0(s)) u_K(s) \right\|_{A(X^0(s))}^2 ds + F(\phi^{u,K}). \end{aligned}$$

Sending $K \rightarrow \infty$ and using Theorem 5.2 gives (5.2), and hence completes the proof of the lower bound (2.13).

References

- [1] R. Azencott and G. Ruget. Mélanges d'équations différentielles et grand écart à la loi des grandes nombres. *Z. Wahrsch. Verw. Gebiete*, 38:1–54, 1977.
- [2] P. Baldi. Large deviations and stochastic homogenization. *Ann. Mat. Pura Appl.*, 151:161–177, 1988.
- [3] J.H. Blanchet, P. Glynn, and K. Leder. On Lyapunov inequalities and subsolutions for efficient importance sampling. *TOMACS*, page to appear, 2012.
- [4] X. Chen and A. de Acosta. Moderate deviations for empirical measures of markov chains: Upper bounds. *J. Theor. Probab.*, 11:1075–1110, 1998.
- [5] A. de Acosta. Moderate deviations for empirical measures of markov chains: Lower bounds. *Ann. Probab.*, 25:259–284, 1997.
- [6] T. Dean and P. Dupuis. Splitting for rare event simulation: A large deviations approach to design and analysis. *Stoch. Proc. Appl.*, 119:562–587, 2009.
- [7] A. Dembo. Moderate deviations for martingales with bounded jumps. *Elect. Comm. Probab.*, 1:11–17, 1996.
- [8] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications*. Jones and Bartlett, Boston, 1993.
- [9] H. Djellout. Moderate deviations for martingale differences and applications to ϕ -mixing sequences. *Stoch. Stoch. Rep.*, 73:37–63, 2002.
- [10] P. Dupuis and R. S. Ellis. *A Weak Convergence Approach to the Theory of Large Deviations*. John Wiley & Sons, New York, 1997.
- [11] P. Dupuis and H. Wang. Importance sampling, large deviations, and differential games. *Stoch. and Stoch. Reports.*, 76:481–508, 2004.
- [12] P. Dupuis and H. Wang. Subsolutions of an Isaacs equation and efficient schemes for importance sampling. *Math. Oper. Res.*, 32:1–35, 2007.
- [13] M. I. Freidlin and A. D. Wentzell. *Random Perturbations of Dynamical Systems*. Springer-Verlag, New York, 1984.

- [14] F.Q. Gao. Moderate deviations for martingales and mixing random processes. *Stoch. Proc. Appl.*, 61:263–275, 1996.
- [15] J. Gärtner. On large deviations from the invariant measure. *Theory Probab. Appl.*, 22:24–39, 1977.
- [16] A. Guillin. Moderate deviations of inhomogeneous functionals of markov processes and application to averaging. *Stoch. Proc. Appl.*, 92:287–313, 2001.
- [17] A. Guillin. Averaging principle of sde with small diffusion: Moderate deviations. *Ann. Probab.*, 31:413–443, 2003.
- [18] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, New York, New York, 1991.
- [19] A.D. Wentzell. Rough limit theorems on large deviations for Markov stochastic processes, I. *Theory Probab. Appl.*, 21:227–242, 1976.
- [20] A.D. Wentzell. Rough limit theorems on large deviations for Markov stochastic processes, II. *Theory Probab. Appl.*, 21:499–512, 1976.
- [21] A.D. Wentzell. Rough limit theorems on large deviations for Markov stochastic processes, III. *Theory Probab. Appl.*, 24:675–692, 1979.
- [22] A.D. Wentzell. Rough limit theorems on large deviations for Markov stochastic processes, IV. *Theory Probab. Appl.*, 27:215–234, 1982.